Spin, Statistics, and Reflections
II. Lorentz Invariance[*]

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Abstract

The analysis of the relation between modular P_1CT-symmetry — a consequence of the Unruh effect — and Pauli's spin-statistics relation is continued. The result in the predecessor to this article is extended to the Lorentz symmetric situation. A model $\mathbf{G}_L$ of the universal covering $\widetilde{L_+^\uparrow} \cong SL(2,\mathbb{C})$ of the restricted Lorentz group $L_+^\uparrow$ is modelled as a reflection group at the classical level. Based on this picture, a representation of $\mathbf{G}_L$ is constructed from pairs of modular P_1CT-conjugations, and this representation can easily be verified to satisfy the spin-statistics relation.

1 Introduction

The spin-statistics connection and the search for its conceptual roots has been a prominent object of investigation in quantum field theory over decades; we refer to Refs. 18, 15, and 14 for detailed discussions of the literature in this field. Spectacular success has been made in deriving the spin-statistics connection from standard properties of quantum fields, but there has always remained some dissatisfaction because these results did not really dig up the physical roots of the principle. Recently, an angular-momentum additivity condition has been established as sufficient and necessary for Pauli's spin-statistics connection in quantum mechanics [15, 16], but this result does not include quantum fields, which will be of interest here.

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[*]Dedicated to Professor H.-J. Borchers on the occasion of his 80th birthday

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In particular, the analysis was confined to finite-component fields, which entails a strong assumption on the representation of the Lorentz group one wishes to investigate. There are, however, infinite-component quantum fields that are covariant under representations *violating* Pauli's spin-statistics connection [17]. What is more, the confinement to finite-component fields is of a purely technical nature; there is no evident physical motivation except the fact that it is met in practically all applications. So despite the merits of the old results, one agreed that some work remained to be done.

In the 1990s, it was realized by several authors [9, 10, 12, 11] that the spin-statistics connection could be derived from the Unruh effect [20, 1, 2]. This phenomenon, which states that a uniformly accelerated observer experiences the vacuum of a quantum field as a thermal state, has recently been derived from basic stability properties of vacuum states [13].

The Unruh effect, in turn, implies an intrinsic form of P_1CT-symmetry, i.e., covariance under conjugations in charge, time, and *one* spatial direction [10]. These conjugations can be extracted from the algebra of field operators by an elementary intrinsic fashion invented by Tomita and Takesaki [19, 3]. This symmetry is referred to as *modular P_1CT-symmetry*.

In 1994/95, two spin-statistics theorems were obtained by Guido and Longo on the one hand [10] and by one of us on the other [12]. Guido and Longo derived the spin-statistics theorem from the Unruh effect, and their result applies to a large class of quantum field theories, including fields with an infinite number of components and massless fields. On the other hand, the result found in Ref 12 merely assumes modular P_1CT-symmetry in the vacuum sector of a Haag-Kastler theory of observables, and it deduced the spin-statistics connection for massive single-particle states, which give rise to topological charges. The elements of the representation of the (homogeneous) symmetry group are products of two modular conjugations each, which yields an elementary algebraic argument. But massless fields are not included, and the setting prevents the observables to be covariant under more than one representation of the Poincaré group. So one result covers a larger class of fields by making stronger symmetry assumptions, whereas the other one minimizes the symmetry assumptions by considering a smaller class of fields.

The result presented in this article joins the advantages of these two approaches: Only modular P_1CT-symmetry is assumed, and the result covers fields satisfying an absolute minimum of standard assumptions. Not even covariance under a representation of the Lorentz group needs to be assumed from the outset; this representation will be *constructed* from the modular P_1CT-operators.

As an important prerequisite, a model of the universal covering group of the restricted Lorentz group will be constructed first. The model is a kind of a reflection group.

In Ref. 14, a model $\mathbf{G}_R$ of the universal covering group $\widetilde{SO(3)} \cong SU(2)$ has been constructed from pairs of reflections at planes in $\mathbb{R}^3$. Considering a general quantum field theory, it was assumed that modular P_1CT-conjugations existed for all reflections along spacelike vectors in a fixed time-zero plane. This symmetry assumption has been shown to be sufficient to construct a covariant representation of $\mathbf{G}_R$, and it is elementary to see that this representation exhibits the spin-statistics relation.

For the restricted Lorentz group $L_+^\uparrow$, such a representation has been constructed earlier by Buchholz, Dreyer, Florig, and Summers [5, 8]. In a more recent article, Buchholz and Summers have given a much more straightforward proof [6]. The short cut found there was the decisive indication that a similar result could also be obtained for the universal covering $\widetilde{L}_+^\uparrow \cong SL(2,\mathbb{C})$, the goal being the generalization of the first derivation of the spin-statistics theorem already obtained in Ref. [12]. Some of their arguments will play an important role at the end of this paper, where such a generalization is established.

This article will be subdivided as follows. In Section 1.1, some preliminaries will be discussed, in Section 1.2, the construction of the covering group $\mathbf{G}_L$ will be outlined in terms of definitions and statements, which will be proved in Section 2. The construction of $\mathbf{G}_L$ will be applied when proving a most general spin-statistics theorem for relativistic quantum fields, which is done in Section 3. In Section 4, it is shown that modular P_1CT-symmetry implies full PCT-symmetry as well and how the present result is related to the results of Guido and Longo obtained in Ref. 10. The article ends with a conclusion.

1.1 Preliminaries

Let $\mathbb{R}^{1+3}$ be the Minkowski spacetime with three spatial dimensions, denote by $g(\cdot,\cdot)$ its Lorentz metric $(x,y) \mapsto g(x,y) =: xy$, by V_+ the open forward light cone,[1] by M_1^+ the hyperboloid $\{x \in V_+ : x^2 = 1\}$, and by H_1 the spacelike unit hyperboloid $\{x \in \mathbb{R}^{1+3} : x^2 = -1\}$.

The Lorentz group L has four connected components. The connected component $L_+^\uparrow =: L_1$ containing the unit element 1 is a subgroup of L called

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[1]The set $\{x \in \mathbb{R}^{1+s} : x^2 > 0\}$ has two connected components. The open forward lightcone V_+ is the future-directed one with respect to the time orientation of $\mathbb{R}^{1+3}$.

the *restricted Lorentz group.* All $\mu \in L_1$ satisfy $\det \mu = 1$ and $\mu V_+ = V_+$. The fixed-point set $FP(\mu)$ of any $\mu \in L_1$ is a linear subspace of $\mathbb{R}^{1+3}$ with zero, one, two, or four dimensions.

We call $\mu \in L_1$ a *generalized boost* if $FP(\mu)$ contains a two-dimensional spacelike subspace, and we call μ a *generalized rotation* if $FP(\mu)$ contains a two-dimensional timelike subspace. The usual notions of boost and rotation require the choice of a time vector $e_0 \in M_1^+$ and its time-zero plane $e_0^\perp$. A generalized boost μ is a boost with respect to e_0 if $FP(\mu) \subset e_0^\perp$, and it is a rotation if $FP(\mu)^\perp \subset e_0^\perp$. For each generalized rotation or boost μ there is more than one $e_0 \in M_1^+$ with respect to which μ is a rotation or boost, respectively.

Note that the unit element is both a generalized boost and a generalized rotation and that the fixed-point sets of all other generalized rotations and boosts are two-dimensional.

Crucial for the analysis to follow is the fact that each element of L_1 is a concatenation of two orthogonal reflections at two-dimensional spacelike planes.[2] Like in Ref. 14, where the corresponding analysis was carried out for the simpler case of rotational symmetry, a simply connected covering of L_1 will now be constructed by endowing these planes with an orientation.

There are several equivalent and useful descriptions of the set **O** of oriented spacelike planes.

1. *Rindler wedges.* The spacelike complement $S' = \{x \in \mathbb{R} : xs < 0 \text{ for all } s \in S\}$ of a spacelike plane S has two connected components, each of which specifies an orientation on S. These components are wedges and have been named after W. Rindler, who endowed them with a spacetime structure on their own. The geodesic observers in this spacetime structure are those observers in $\mathbb{R}^{1+3}$ that are uniformly accelerated perpendicular to S. The boundary of a Rindler wedge is a horizon for the Rindler observer. This physical role will be relevant in the discussion of the spin-statistics relation below.

2. *Classes of zweibeine.* Define a set of zweibeine Z by

$$Z := \{\xi = (t_\xi, x_\xi) : t_\xi^2 = 1, x_\xi^2 = -1, \ x_\xi \perp t_\xi\}.$$

The set $\xi^\perp := \{t_\xi, x_\xi\}^\perp$ is a two-dimensional spacelike plane, and the wedge $W_\xi := \{x \in \mathbb{R}^{1+3} : xx_\xi > |xt_\xi|\}$ is one of its Rindler wedges. Define an equivalence relation $\xi \tilde{\sim} \eta$ on Z by the condition $W_\xi = W_\eta$.

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[2] see Lemma 13 below.

Let $\bar{Z}$ be the quotient space $Z/\bar{\sim}$, and let $\bar{\pi}$ be the canonical projection from Z onto $\bar{Z}$. For each $a = \bar{\pi}(\xi)$, denote by W_a the wedge W_ξ and by $a^\perp$ its edge $\xi^\perp$.

Given $a \in \bar{Z}$, the hyperbola $\Gamma(a) := \{x_\xi : \xi \in \bar{\pi}^{-1}(a)\}$ is a geodesic of the Rindler spacetime structure on W_a.

An action of the full Lorentz group L on Z from the left is defined by $\mu\xi = (\mu t_\xi, \mu x_\xi)$. Since $\xi \bar{\sim} \eta$ implies $\mu\xi \bar{\sim} \mu\eta$, this action induces an action on $\bar{Z}$ by $\mu\bar{\pi}(\xi) := \bar{\pi}(\mu\xi)$. Evidently, $W_{\mu a} = \mu W_a$.

The subset $Z^+ := \{\xi \in Z : t_\xi \in V_+\}$ of Z has the property that $\bar{\pi}(Z^+) = \bar{\pi}(Z)$. If one restricts the equivalence relation $\bar{\sim}$ to Z^+, one obtains an equivalence relation as well, and the corresponding quotient space is isomorphic with $\bar{Z}$. The restricted Lorentz group L_1 acts transitively on Z^+ (see Lemma 10 below), but not on Z (since the elements of L_1 preserve time orientation).

3. *Spectral decompositions of boosts.* Given $a \in \bar{Z}$, the generalized boosts with fixed-point set $a^\perp$ give rise to a one-parameter group $(\mu_\chi^a)_\chi$ with the property that $(\mu_\chi^a x - x)^2 > 0$ for all $\chi > 0$ and $x \in W_a$. This group is unique up to multiplication of χ by a positive scalar.

Conversely, given a one-parameter group $(\mu_\chi)_{\chi \in \mathbb{R}}$ of generalized boosts, the μ_χ with $\chi \neq 0$ have a common fixed-point plane S. Furthermore, one verifies that there are an $\alpha > 0$ and a future-directed lightlike vector ℓ^+ with $\mu_\chi \ell^+ = e^{\alpha\chi}\ell^+$ for all $\chi \in \mathbb{R}$, and a past-directed lightlike vector ℓ^- with $\mu_\chi \ell^- = e^{-\alpha\chi}\ell^-$ for all $\chi \in \mathbb{R}$. The convex hull of ℓ^+, ℓ^-, and S is the closure of a Rindler wedge.

Pairs of lightlike vectors have been used earlier by Buchholz, Dreyer, Florig, and Summers for the description of Rindler wedges [5].

The subsequent analysis will be formulated in terms of $\bar{Z}$ rather than the naturally isomorphic set **O**, but occasionally, the other descriptions show up in the proofs as well.

1.2 The construction of $\mathbf{G}_L$: definitions and statements

For each $\xi \in Z$, let both j_ξ and $j_{\bar{\pi}(\xi)}$ denote the orthogonal reflection by the plane $\xi^\perp = a^\perp$, i.e., the map

$$j_\xi x \equiv j_{\bar{\pi}(\xi)} x := x - 2(x t_\xi)\, t_\xi - 2(x x_\xi)\, x_\xi.$$

Endow the set $\bar{Z} \times \bar{Z} =: \mathbf{M}_L$ with the structure of the pair groupoid of $\bar{Z}$, and define an operation of L on $\mathbf{M}_L$ by $\mu(a,b) := (\mu a, \mu b)$. With each $(a,b) \in \mathbf{M}_L$, one can associate the Lorentz transformation $\lambda(a,b) := j_a j_b \in L_1$. Define a relation $\sim$ on $\mathbf{M}_L$ by writing $\underline{m} \sim \underline{n}$ if and only if there exists a $\mu \in L_1$ with $\underline{n} = \pm\mu^2 \underline{m}$ and $\mu\lambda(\underline{m})\mu^{-1} = \lambda(\underline{m})$. Note that $\mu \sim \nu$ implies $\lambda(\mu) = \lambda(\nu)$.[3]

Proposition 1. *The relation $\sim$ is an equivalence relation.*

The proof will be given in section 2.2.

Let $\mathbf{G}_L$ be the quotient space $\mathbf{M}_L/\sim$, and denote by $\pi : \mathbf{M}_L \to \mathbf{G}_L$ the canonical projection of the relation $\sim$. Define $\pm 1 := \pi(a, \pm a)$ for arbitrary $a \in \bar{Z}$, and $-\pi(a,b) := \pi(a,-b)$ for $(a,b) \in \mathbf{M}_L$.

Proposition 2. *For each $g \in \mathbf{G}_L$, one has $g \neq -g$ and $\tilde{\lambda}^{-1}(\tilde{\lambda}(g)) = \{g, -g\}$.*

The proof of this theorem will be given in Section 2.3.

As remarked, $\mu \sim \nu$ implies $\lambda(\mu) = \lambda(\nu)$, so a map $\tilde{\lambda} : \mathbf{G}_L \to L_1$ can be defined by $\tilde{\lambda}(g) := \lambda(\pi^{-1}(g))$, and the diagram

$$
\begin{array}{ccc}
\mathbf{M}_L & \xrightarrow{\;\pi\;} & \mathbf{G}_L \\
{\scriptstyle \lambda}\big\downarrow & \swarrow {\scriptstyle \tilde{\lambda}} & \\
L_+^\uparrow & &
\end{array}
\qquad (1)
$$

commutes. All maps in this diagram are continuous. This holds for π by definition, and it is evident for λ. To show continuity of $\tilde{\lambda}$, let $M \subset L_1$ be open. $\tilde{\lambda}^{-1}(M)$ is open if and only if $\pi^{-1}(\tilde{\lambda}^{-1}(M))$ is open. This set coincides with $\lambda^{-1}(M)$, which is open by continuity of λ.

Theorem 3.

(i) *$\tilde{\lambda}$ is a covering map and endows $\mathbf{G}_L$ with the structure of a two-sheeted covering space of L_1.*

(ii) *$\mathbf{G}_L$ is simply connected.*

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[3]The square in the condition $\underline{n} = \pm\mu^2 \underline{m}$ is important in order to avoid trouble with rotations by the angle π. It has been forgotten in Ref. [14].

(iii) There is a unique group product $\odot$ on $\mathbf{G}_L$ with the property that the diagram

$$
\begin{array}{ccc}
\mathbf{M}_L \times \mathbf{M}_L & \xrightarrow{\ \circ\ } & \mathbf{M}_L \\
{\scriptstyle \pi \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathbf{G}_L \times \mathbf{G}_L & \xrightarrow{\ \odot\ } & \mathbf{G}_L \\
{\scriptstyle \tilde{\lambda} \times \tilde{\lambda}}\downarrow & & \downarrow{\scriptstyle \tilde{\lambda}} \\
L_1 \times L_1 & \xrightarrow{\quad \cdot \quad} & L_1
\end{array}
\tag{2}
$$

commutes.

This means that $\mathbf{G}_L$ is isomorphic with the universal covering group of L_1. The proof will be given in section 2.6.

Lemma 4 (Adjoint action of $\mathbf{G}_L$ on itself). *Given $h \in \mathbf{G}_L$ and $(c,d) \in \mathbf{M}_L$, one has*

$$
h\pi(c,d)h^{-1} = \pi\left(\tilde{\lambda}(h)c, \tilde{\lambda}(h)d\right). \tag{3}
$$

2 The construction of $\mathbf{G}_L$: proofs

The proofs of the statements made in the preceding section requires an extended mathematical analysis, which will now be developed step by step.

2.1 Reflections of spacelike planes by spacelike planes

It may well be that the following lemma, which is highly plausible at a first glance, but somewhat tricky to prove, has been proved earlier by other authors. But since such a reference is not known to us, we prove it here.

Lemma 5. *If A and B are two-dimensional spacelike planes of $\mathbb{R}^{1+3}$, then there exists a spacelike plane C such that B is the image of A under orthogonal reflection by C.*

Proof. If A and B have nontrivial intersection, then there exist linearly independent nonzero vectors $a \in A$, $b \in B$, and $c \in A \cap B$. The one-dimensional timelike space $\{a,b,c\}^{\perp}$ is perpendicular to both A and B, so A and B are subspaces of a common time-zero plane, and the problem boils down to the well-known three-dimensional euclidean case.

It remains to consider the case that A and B have trivial intersection. $A^{\perp}$ and $B^{\perp}$ are timelike planes and, hence, are spanned by future-directed

lightlike vectors $x, y \in A^\perp$ and $v, w \in B^\perp$. Since A and B have trivial intersection, x, y, v, and w are linearly independent.

Let C be the plane spanned by the vectors $x - \alpha v$ and $y - \beta w$, where

$$\alpha := \sqrt{\frac{xy}{vw}\frac{xw}{yv}} > 0 \qquad \text{and} \qquad \beta := \sqrt{\frac{xy}{vw}\frac{yv}{xw}} > 0.$$

Then $C^\perp$ is spanned by $x + \alpha v$ and $y + \beta w$, since

$$(x - \alpha v)(x + \alpha v) = x^2 - \alpha^2 v^2 = 0 = (y - \beta w)(y + \beta w)$$

and since

$$(x - \alpha v)(y + \beta w) = xy - \alpha\beta\, vw - \alpha\, yv + \beta\, xw$$

$$= xy - \sqrt{\frac{xy}{vw}\frac{xw}{yv}\frac{xy}{vw}\frac{yv}{xw}}\, vw - \sqrt{\frac{xy}{vw}\frac{xw}{yv}}\, yv + \sqrt{\frac{xy}{vw}\frac{yv}{xw}}\, xw$$

$$= xy - \frac{xv}{vw}\, vw - \sqrt{\frac{xy}{vw}\,(xw)(yv)} + \sqrt{\frac{xy}{vw}\,(yv)(xw)}$$

$$= 0 = \cdots = (x + \alpha v)(y - \beta w).$$

$C^\perp$ is timelike because

$$(x + \alpha v)^2 = x^2 + 2\alpha\, xv + \alpha^2 v^2 = 2\alpha\, xv > 0$$

by x and v beeing future directed and by $\alpha > 0$, so C is spacelike. Denote by j_C the orthogonal reflection at C. One then finds

$$j_C x = \tfrac{1}{2} j_C \left(\underbrace{(x + \alpha v)}_{\in C^\perp} + \underbrace{(x - \alpha v)}_{\in C} \right) = \tfrac{1}{2}\left(-(x + \alpha v) + (x - \alpha v)\right) = -\alpha v \in B$$

and $j_C y = -\beta w \in B$. $\qquad\qquad\square$

2.2   Proof of Proposition 1

Proposition 1 states that the relation $\sim$ is an equivalence relation. In contrast to the corresponding statement for the analysis of the rotation group and its universal covering, this is not self-evident. It will be proved in this section, together with some properties of the equivalence relation $\sim$.

Lemma 6. *Let μ and ν be restricted Lorentz transformations.*

(i) *Suppose that $\mu \neq 1$. There exist at least one and at most two elements ν with $\mu = \nu^2$. If, in particular, $\mu^2 = 1$, there are two such square roots ν_+ and ν_-, and $\nu_+\nu_- = 1$.*

(ii) *The commutant of μ is an abelian group if and only if $\mu^2 \neq 1$.*

(iii) *Given $\mu, \nu \in L_1$, suppose that $\mu^2 \neq 1 \neq \nu^2$ and $\mu^2\nu^2 = \nu^2\mu^2$. Then $\mu\nu = \nu\mu$.*

Proof. The matrix group $SL(2,\mathbb{C})$ is well known to be isomorphic with the universal covering group of L_1. Let Λ be any covering map from $SL(2,\mathbb{C})$ onto L_1. Then $\Lambda^{-1}(\Lambda(A)) = \pm A$ for any $A \in SL(2,\mathbb{C})$.

The conjugacy classes of $SL(2,\mathbb{C})$ are classified by the Jordan matrices in $SL(2,\mathbb{C})$, which are

$$N_z := \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}, z \in \dot{\mathbb{C}} \qquad N_\infty := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad N_{-\infty} := \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

so for each $A \in SL(2,\mathbb{C})$ there is a $z \in \dot{\mathbb{C}} \cup \pm\infty$ and a $P \in SL(2,\mathbb{C})$ with $A = PN_zP^{-1}$.

Proof of (i). Since $\mu \neq 1$ by assumption and since $[N_{-z}] = [-N_z]$, there exists an element $A = PN_zP^{-1} \in \Lambda^{-1}(\mu)$ with $\pm 1 \neq z \neq -\infty$.

If $z \neq \infty$, the elements of $\Lambda^{-1}(\mu)$ are $\pm A$, and $B_\pm \in SL(2,\mathbb{C})$ satisfy $B_\pm^2 = \pm A$ if and only if $\pm B_\pm = \pm PN_{w_\pm}P^{-1}$ for complex square roots $w_\pm$ of $\pm z$. One obtains two square roots $\nu_\pm := \Lambda(B_\pm) \equiv \Lambda(-B_\pm)$ of μ.

If $z = \infty$, then $B := \pm P\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}P^{-1}$ are the two square roots of A. Since the elements in $[N_{-\infty}]$ have no square roots in $SL(2,\mathbb{C})$, the only square root of μ is $\nu := \Lambda(B) \equiv \Lambda(-B)$.

If $\mu^2 = 1$, then $z^2 = \pm i$, so μ has two roots ν_+ and ν_-. In order to prove $\nu_+\nu_- = 1$, let w_+ be a square root of i, then $w_- := \overline{w_+}$ is a square root of $-i$. One obtains

$$\nu_+\nu_- = \Lambda(PN_{w_+}P^{-1})\Lambda(PN_{w_-}P^{-1}) = \Lambda(PN_{w_+}P^{-1}PN_{\overline{w_+}}P^{-1}) = 1.$$

Proof of (ii). $\mu\nu = \nu\mu$ if and only if $AB = \pm BA$ for all $A \in \Lambda^{-1}(\mu)$ and $B \in \Lambda^{-1}\nu$.

Given $A = PN_zP^{-1} \in SL(2,\mathbb{C})$ with $z \neq \pm 1$, the commutant of A is the abelian group $\{PN_zP^{-1} : z \in \dot{\mathbb{C}}\}$.

The anticommutant of A is trivial if $z \neq \pm i$; otherwise it consists of the matrices $P\begin{pmatrix} 0 & v \\ -1/v & 0 \end{pmatrix}P^{-1}$. These matrices neither commute nor anticommute with the elements of the commutant of $PN_{\pm i}P^{-1}$.

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But if $\mu^2 \neq 1$, then there exists an $A = PN_zP^{-1} \in \Lambda^{-1}(\mu^2)$ with $\pm 1 \neq z \neq \pm i$}, so the commutant A^c of A is an abelian subgroup of $SL(2, \mathbb{C})$, and the anticommutant of A is trivial. Accordingly, the commutant μ^c of μ is the abelian group $\Lambda(A^c)$.

If $\mu^2 = 1$, all $z \in \mathbb{C}$ with $A = PN_zP^{-1}$ and $\Lambda(A) = \mu$ equal ± 1 or $\pm i$. If $z = \pm 1$, then $\mu = 1$, and the commutant is L_1 and, hence, nonabelian, and if $z = \pm i$, the above remarks apply.

Proof of (iii). Since $\mu^2 \neq 1 \neq \nu^2$, it follows from the preceding statement that the commutants μ^c and ν^c are the maximal-abelian groups $\{PN_zP^{-1} : z \in \dot{\mathbb{C}}\}$, $\{PN_zP^{-1} : z \in \dot{\mathbb{C}}\}$, or $\left\{P\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}P^{-1} : t \in \mathbb{C}\right\}$. for some $P \in SL(2, \mathbb{C})$. Consequently, the assumption $\nu^2 \in (\mu^2)^c$ implies $\nu \in (\mu^2)^c$, i.e., $\mu^2 \in (\nu^2)^c$. This yields the statement by the same argument. $\quad\square$

Lemma 7. *Consider $(a, b) \in \mathbf{M}_L$, and shorthand $\lambda(a, b) =: \lambda$.*
There exists an element $c \in \bar{Z}$ with $a = \pm j_c b$. Shorthanding $\lambda(c, b) =: \mu$, one has $\mu^2 = \lambda$ and $(a, b) = (\mp \mu b, b)$.

Proof. Existence of c follows from Lemma 5. The other statements follow from

$$j_c j_b j_c j_b = j_{j_c b} j_b = j_a j_b$$

and

$$a = \pm j_c b = \mp j_c j_b b = \mp \mu b. \qquad\qquad \square$$

Proposition 1. $\sim$ *is an equivalence relation.*

Proof. Symmetry and reflexivity are evident, so it remains to prove transitivity. If $\underline{m} \sim \underline{n}$ and $\underline{n} \sim \underline{r}$, then $\lambda(\underline{m}) = \lambda(\underline{n}) = \lambda(\underline{r}) =: \lambda$, and there exist elements μ and ν commuting with λ and satisfying $\mu^2 \underline{m} = \pm \underline{n}$ and $\nu^2 \underline{n} = \pm \underline{r}$. If $\mu^2 = 1$ or $\nu^2 = 1$, one trivially has $\underline{m} \sim \underline{r}$. If $\nu^2 \mu^2 = 1$, one even has $\underline{m} = \pm \underline{r}$. It follows from $\nu^2 \mu^2 \underline{m} = \pm \underline{r}$ that

$$\lambda = j_{\nu^2 \mu^2 a} j_{\nu^2 \mu^2 b} = \nu^2 \mu^2 j_a j_b \mu^{-2} \nu^{-2} = \nu^2 \mu^2 \lambda \mu^{-2} \nu^{-2},$$

and one concludes from Lemma 6.ii that there exists a square root κ of $\nu^2 \mu^2$ commuting with λ. $\quad\square$

2.3 Proof of Proposition 2

Lemma 8. $(a, b) \not\sim (a, -b)$ *for* $(a, b) \in \mathbf{M}_L$, *i.e.,* $g \neq -g$ *for all* $g \in \mathbf{G}_L$.

Proof. The statement is evident for $b = \pm a$, so it remains to consider the case $\lambda(a,b) \neq 1$.

Assume $(a,b) \sim (a,-b)$. By Lemma 7, there exists an element $\mu \in L_1$ with $\mu^2 = \lambda(a,b)$ and $(a,b) = (\pm\mu b, b)$, and by assumption, there exists an element $\nu \in L_1$ with $\nu\mu^2\nu^{-1} = \mu^2$ and $\nu^2(a,b) = \pm(a,-b)$. μ^2 and ν^2 commute and differ from 1, so μ and ν commute by Lemma 6.iii. Assume without loss that $a = \mu b$, then one obtains

$$(a,b) = (\mu b, b) = \pm\nu^2(\mu b, -b) = \pm(\mu\nu^2 b, -\nu^2 b) = \pm(-\mu b, b) = \pm(-a, b),$$

leading to the contradiction $a = -a$ or $b = -b$, respectively. □

Lemma 9. *Suppose that $\lambda(a,b) = \lambda(c,d)$. Then*

(i) $\lambda(a,c) = \lambda(b,d)$.

(ii) $\lambda(a,b)$ *and* $\lambda(a,c)$ *commute.*

(iii) $(a,b) \sim (c,d)$ *or* $(a,b) \sim (c,-d)$.

Proof of (i). $j_a j_c = j_a(j_c j_d)j_d = j_a(j_a j_b)j_d = j_b j_d$.

Proof of (ii). $j_a j_b j_c j_a j_b j_a = (j_a j_b)j_c(j_a j_b)j_a = j_c j_d j_c j_c j_d j_a = j_c j_a$.

Proof of (iii). Since by definition $(a,b) \sim (-a,-b)$, it suffices to prove $(a,b) \sim (\pm c, \pm d)$ for an arbitrary choice of signs. If $\lambda(a,b) = 1$ or $\lambda(c,a) = 1$ the statement is trivial. So assume $\lambda(a,b) \neq 1 \neq \lambda(c,a)$.

By Lemma 7, there exist square roots ν_{ab} and ν_{cd} of $\lambda(a,b)$ and square roots ν_{ca} and ν_{db} of $\lambda(c,a)$ with

$$a = \pm\nu_{ab}b, \quad c = \pm\nu_{cd}d \quad \text{and} \quad c = \pm\nu_{ca}a, \quad d = \pm\nu_{db}b$$

for some choice of signs.

It suffices to prove $\nu_{cd}\nu_{db} = \nu_{db}\nu_{cd}$ and $\nu_{ab}b = \pm\nu_{cd}b$, since these relations yield the statement by

$$(c,d) = (\pm\nu_{cd}\nu_{db}b, \pm\nu_{db}b) = \nu_{db}(\pm\nu_{cd}b, \pm b) = \nu_{db}(\pm\nu_{ab}b, \pm b) = \nu_{db}(\pm a, \pm b).$$

If $\lambda(a,b)^2 \neq 1$, one obtains $\nu_{cd}\nu_{db} = \nu_{db}\nu_{cd}$ from statement (ii) and Lemma 6 (iii). The remaining condition $\nu_{ab}b = \pm\nu_{cd}b$ then follows from

$$j_{\nu_{ab}b} = j_c(j_c j_a) = j_c(j_d j_b) = j_c(\nu_{cd}(j_d j_b)\nu_{cd}^{-1}) = j_c(j_c j_{\nu_{cd}b})) = j_{\nu_{cd}b}.$$

If $\lambda(a,b)^2 = 1$, one obtains $a = \pm\lambda(a,b)b$ from $1 = j_a j_b j_a j_b = j_a j_{-j_b a}$. Lemma 6 (i) implies $\nu_{ab}^{-1}\nu_{cd} = \lambda(a,b)$ or $\nu_{ab}^{-1}\nu_{cd} = 1$, proving $\nu_{ab}b = \pm\nu_{cd}b$. The proof is completed by

$$\nu_{cd}\lambda(d,b)\nu_{cd}^{-1} = j_c j_{\nu_{cd}b} = j_c j_{\nu_{ab}b} = j_c j_a = \lambda(d,b)$$

and an application of Lemma 6 (iv) yielding $\nu_{cd}\nu_{db} = \nu_{db}\nu_{cd}$. □

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One now immediately obtains

Proposition 2. *For each $g \in \mathbf{G}_L$ the fiber $\tilde{\lambda}^{-1}(\tilde{\lambda}(g))$ contains precisely two elements.*

Proof. $g \neq -g$ and $\tilde{\lambda}(g) = \tilde{\lambda}(-g)$ for all g, so each $\tilde{\lambda}^{-1}(\tilde{\lambda}(g))$ contains at least two elements.

By construction, one has $\lambda(a,b) = \tilde{\lambda}(g)$ for each $(a,b) \in \pi^{-1}(g)$. If $(c,d) \in \mathbf{M}_L$ satisfies $\lambda(c,d) = \tilde{\lambda}(g) = \lambda(a,b)$ as well, Lemma 9 implies that $(a,b) \sim (c,d)$ or $(a,b) \sim (c,-d)$, so $\tilde{\lambda}^{-1}(\tilde{\lambda}(g))$ contains at most two elements. $\square$

2.4 The sets Z and $\bar{Z}$

The next goal is the proof of Theorem 3. This requires some preliminaries. Some properties of the sets Z and $\bar{Z}$ will be worked out in this section, and the polar decomposition of restricted Lorentz transformations into rotations and boosts will be discussed in the next section.

For each $x \in \mathbb{R}^{1+3}$, denote the stabilizer of x in L_1 as $\mathfrak{S}(x) := \{\mu \in L_1 : \mu x = x\}$, and for each subset M of $\mathbb{R}^{1+3}$, define $\mathfrak{S}(M) := \bigcap_{x \in M} \mathfrak{S}(x)$.

Lemma 10. *The actions of L_1 on Z^+ and of L on Z are transitive.*

Proof. Consider any $\xi, \eta \in Z^+$. M_1^+ is an orbit of L_1, so there exists a Lorentz transformation μ with $t_\xi = \mu t_\eta$. This μ is not unique, since $t_\xi = \nu \mu t_\eta$ for each $\nu \in \mathfrak{S}(t_\xi)$.

By construction, one already has $\mu x_\eta \perp t_\xi$, so it remains to be shown that $\mathfrak{S}(t_\xi)$ acts transitively on $H_1 \cap \{t_\xi\}^\perp$. But $\mathfrak{S}(t_\xi)$ is the group of rotations with respect to the time vector t_ξ, and $H_1 \cap \{t_\xi\}^\perp$ is the set of time-zero unit vectors, on which $\mathfrak{S}(t_\xi)$ acts transitively.

The second statement now follows from the fact that $-1 \in L$. $\square$

Lemma 11. *$\bar{Z}$ is a first-countable topological space.*

Proof. Let H be a Cauchy surface. Then the set $Z_H := \{\xi \in Z^+ : x_\xi \in H\}$ is a closed subset of Z^+.

For each $\xi \in Z^+$, the intersection of the inextendible curve $\Gamma(\xi)$ with H contains precisely one element y_ξ, and there is a unique generalized boost $\beta_H(\xi)$ with $y_\xi = \beta_H(\xi) x_\xi$.

Define a map $\zeta_H : Z^+ \to Z_H$ by $\zeta_H(\xi) := \beta_H(\xi)\xi$. Then $\xi \underset{\sim}{\approx} \eta$ implies $\zeta_H(\xi) = \zeta_H(\eta)$ by construction, so a map $\bar{\zeta} : \bar{Z} \to Z_H$ is well defined by

$\bar{\zeta}(\bar{\pi}(\xi)) = \zeta(\xi)$. The diagram

$$
\begin{array}{ccc}
Z^+ & \xrightarrow{\bar{\pi}} & \bar{Z} \\
\zeta_H \downarrow & \swarrow \bar{\zeta}_H & \\
Z_H & &
\end{array}
$$

commutes. All maps in this diagram are continuous. This holds for $\bar{\pi}$ by definition, and it is evident for ζ_H. To show continuity of $\bar{\zeta}_H$, let $M \subset L_1$ be open. $\bar{\zeta}_H^{-1}(M)$ is open if and only if $\pi^{-1}(\bar{\zeta}_H^{-1}(M))$ is open. This set coincides with $\zeta_H^{-1}(M)$, which is open by continuity of ζ_H.

Since $\bar{\zeta}_H$ has the continuous inverse $\bar{\pi}|_{Z_H}$, one finds that Z_H and $\bar{Z}$ are homeomorphic topological spaces. Since Z_H is first-countable, so is $\bar{Z}$. $\quad\square$

One immediately concludes the following corollary.

Corollary 12. $\bar{Z}$ *and* $\mathbf{M}_L$ *are Hausdorff spaces.*

2.5 Polar decompositions on L_1

The next task will be the proof of Theorem 3, which, again, is much more involved than its prototype in Ref. 14. A crucial instrument will be the decomposition of Lorentz transformations into rotations and boosts. Specify a time direction by distinguishing a future-directed timelike unit vector e_0.

Consider the euclidean inner product $\langle \cdot, \cdot \rangle_{e_0}$ on $\mathbb{R}^{1+3}$ defined by $\langle x, y \rangle_{e_0} := -g(x, y) + 2g(x, e_0)g(y, e_0)$. Denote the adjoint of a linear map $T : \mathbb{R}^{1+3} \to \mathbb{R}^{1+3}$ with respect to this inner product by T^*. If T is an automorphism, then the positive operator $\hat{\beta}(T) := |T| := (T^*T)^{1/2}$ is a boost, and the orthogonal operator $\hat{\rho} := T \cdot |T|^{-1} = T\hat{\beta}(T)^{-1}$ is a rotation; $\hat{\beta}(T)$ and $\hat{\rho}(T)$ yield the polar decomposition $T = \hat{\rho}(T)\hat{\beta}(T)$ of T. On $\mathbf{G}_L$, define $\tilde{\rho}(g) := \hat{\rho}(\tilde{\lambda}(g))$ and $\tilde{\beta}(g) := \hat{\beta}(\tilde{\lambda}(g))$.

To each time-zero unit vector e, assign the class $\bar{e} := \bar{\pi}(e_0, e)$. The following lemma immediately follows from Lemma 2.1 in Ref. 6; the proof is recalled here for the reader's convenience.

Lemma 13. $\tilde{\lambda}$ *is onto.*

Proof. We prove that λ is onto, then the statement follows. $\lambda(a, \pm a) = 1$ for all $a \in \bar{Z}$, so it remains to show that $\lambda^{-1}(\mu) \neq \emptyset$ for each $\mu \neq 1$.

Suppose that $\mu =: \rho$ is a rotation, that τ is a root of ρ, and that e is a time-zero unit vector in the rotation plane of ρ. Then $\rho = \rho j_{\bar{e}} j_{\bar{e}} = j_{\tau \bar{e}} j_{\bar{e}} = \lambda(\tau \bar{e}, \bar{e})$.

Suppose that $\mu =: \beta$ is a boost, and let e be a time-zero unit vector in the fixed-point set of β. Then $\beta = j_{\bar{e}} j_{\bar{e}} \beta = j_{\bar{e}} j_{\beta^{-1/2} \bar{e}} = \lambda(\bar{e}, \beta^{-1/2} \bar{e})$.

In the remaining case that both $\hat{\rho}(\mu)$ and $\hat{\beta}(\mu)$ differ from 1, the rotation plane of $\hat{\rho}(\mu)$ and the fixed-point plane of $\hat{\beta}(\mu)$ are well-defined two-dimensional planes contained in the time-zero plane. Since the time-zero plane is three-dimensional, this implies that the intersection of these planes is nonempty. Let e be a unit vector in this intersection and let τ be a root of $\hat{\rho}(\mu)$. Then $\mu = \hat{\rho}(\mu) j_{\bar{e}} j_{\bar{e}} \hat{\beta}(\mu) = j_{\tau \bar{e}} j_{\hat{\beta}(\mu)^{-1/2} \bar{e}} = \lambda(\tau \bar{e}, \beta^{-1/2}(\mu) \bar{e})$. $\qquad \square$

Define $\dot{R} := R \backslash \{1\}$ and $\dot{B} := B \backslash \{1\}$, and write $\ddot{R} := \{\sigma \in R : \sigma^2 \neq 1\}$.

Lemma 14. $\rho \in \dot{R}$ and $\beta \in \dot{B}$ commute if and only if $FP(\rho) = FP(\beta)^{\perp}$.

Proof. Assume $\rho\beta = \beta\rho$. If $x \in FP(\beta)$, then $\beta\rho x = \rho\beta x = \rho x$, so $\rho FP(\beta) = FP(\beta)$, whence one concludes that either $FP(\beta) = FP(\rho)$ or $FP(\beta) = FP(\rho)^{\perp}$. Since $FP(\beta)$ is a spacelike surface, whereas $FP(\rho)$ is timelike, one concludes $FP(\beta) = FP(\rho)^{\perp}$. That the condition is sufficient, is trivial. $\quad\square$

Lemma 15.

(i) *Consider $\mu \in L_1$ with polar decomposition $\mu = \rho\beta$. Then $\rho\beta = \beta\rho$ if and only if there exists a time-zero unit vector e with $\mu \in \mathfrak{S}(\bar{e})$.*

(ii) *Given $a, b \in \bar{Z}$, one has $\mathfrak{S}(a) \cap \mathfrak{S}(b) \neq \{1\}$ if and only if $a = \pm b$.*

Proof of (i). Each rotation or boost is contained in the stabilizer of $\bar{e}$ for some e, so statement (i) trivially holds for rotation or boosts.

It remains to consider the case that $\rho \neq 1 \neq \beta$. If $\rho\beta = \beta\rho$, then it follows from Lemma 14 that the rotation axis of ρ is parallel to the boost direction of β. Let e be one of the two unit vectors on this axis, then ρ, β, and, hence, also $\rho\beta$ are contained in $\mathfrak{S}(W_{\bar{e}}) = \mathfrak{S}(\bar{e})$. So the condition is necessary.

If, conversely, $\mu \in \mathfrak{S}(\bar{e})$, then there exists a unique boost γ with $\gamma(\mu e_0, \mu e) = (e_0, e)$, and $\gamma \in \mathfrak{S}(\bar{e})$ because $\gamma \bar{e} = \gamma\mu\bar{e} = \bar{e}$. Because $\mathfrak{S}(\bar{e})$ is abelian, $\gamma\mu = \mu\gamma = \rho\beta\gamma$.

The product $\rho\beta\gamma$ has the fixed points e_0 and e by definition of γ, so it is a rotation, and $\beta\gamma = 1$ by uniqueness of the polar decomposition. As seen above, γ commutes with μ, so β^{-1} commutes with $\rho\beta$, i.e., $\rho = \beta^{-1}\rho\beta$.

Proof of (ii). Without loss, assume $a = \bar{e}$. If μ is a rotation, then e is on the rotation axis of μ, so $FP(\mu) = \bar{e}^{\perp\perp}$, and the plane $\bar{e}^{\perp}$ is mapped onto itself. The only other time-zero unit vector on the axis of μ is $-e$, so $b = \pm\bar{e} = \pm a$, as stated.

If μ is a boost, then the vectors $\ell^{+} := e + e_0$ and $\ell^{-} := e - e_0$ are eigenvectors of μ associated with distinct eigenvalues ε and ε^{-1}. The vectors $\ell^{\pm}$ are perpendicular to $FP(\mu)$ by invariance of the metric: if $x \in FP(\mu)$, then

$$\varepsilon \, g(x, \ell^{+}) = g(x, \mu\ell^{+}) = g(\mu^{-1}x, \ell^{+}) = g(x, \ell^{+}),$$

so $\varepsilon \neq 1$ implies $g(x, \ell^{+}) = 0$, and one obtains $FP(\mu) = \bar{e}^{\perp}$.

It remains to consider the case that $\rho \neq 1 \neq \beta$. By Lemma 14, statement (i) implies $FP(\rho) \perp FP(\beta)$, so $\ell^{\pm}$ are fixed points of ρ and, hence eigenvectors not only of β, but also of μ. Additional eigenvectors in $\bar{e}^{\perp}$ exist only if ρ is a rotation by the angle π; their eigenvalue is -1. Since $\varepsilon \neq -1 \neq \varepsilon^{-1}$, the vectors $\ell^{\pm}$ are the only eigenvectors of μ with positive eigenvalues.

By assumption, $\mu \in \mathfrak{S}(b) =: \mathfrak{S}(\bar{\pi}(f_0, f))$, so the polar decomposition of μ with respect to f_0 commutes. The reasoning just used yields that the lightlike vectors $f + f_0$ and $f - f_0$ are eigenvectors of μ with positive eigenvalues and, hence, proportional to $e+e_0$ and $e-e_0$, respectively, whence $\bar{e} = \pm\bar{f}$ and, hence, statement (ii) is obtained. □

Lemma 16. *Given any $\mu \in L_1$, suppose that the polar decomposition $\mu = \rho_{e_0}\beta_{e_0}$ commutes for all $e_0 \in M_1^{+}$. Then $\mu = 1$.*

Proof. Because by assumption, $\rho_{e_0}\beta_{e_0} = \beta_{e_0}\rho_{e_0}$, there is some time-zero unit vector e with $\mu \in \mathfrak{S}(\bar{e})$.

The subset

$$t_{\bar{e}} := \{d_0 \in M_1^{+} : \bar{\pi}(d_0, d) = \bar{e} \quad \text{for some unit vector } d \perp d_0\}$$

of M_1^{+} is a hyperbola, so there exists some $f_0 \in M_1^{+}\backslash t_{\bar{e}}$.

By assumption, the polar decomposition $\mu = \rho_{f_0}\beta_{f_0}$ commutes as well, so there is some unit vector $f \perp f_0$ with $\mu \in \mathfrak{S}(\bar{\pi}(f_0, f))$. By construction, $\bar{\pi}(f_0, f) \neq \pm\bar{e}$, so $W_{\bar{\pi}(f_0,f)} \neq \pm W_{\bar{e}}$, whence $\mathfrak{S}(\bar{\pi}(f_0, f)) \cap \mathfrak{S}(\bar{e}) = \{1\}$ by Lemma 15. □

For each $(\rho, \beta) \in \dot{R} \times B$, let $E(\rho, \beta)$ be the set of all time-zero unit vectors in $FP(\rho)^{\perp} \cap FP(\beta)$.

Proposition 17.

(i) $E(\rho, \beta) \cong S^1$ *if and only if* $\rho\beta = \beta\rho$.

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(ii) Otherwise, $E(\rho, \beta) = \{\pm e\}$ for some time-zero unit vector e.

Proof of (i). If $\beta = 1$, then $E(\rho, \beta) = FP(\rho)^{\perp} \cap \big(\{0\} \times S^2\big)$, i.e., the intersection of the time-zero two-sphere with a two-dimensional spacelike subspace of the time-zero plane. Such an intersection is homeomorphic to S^1. If $\rho \neq 1 \neq \beta$, then $\rho\beta = \beta\rho$ if and only if $FP(\beta) \perp FP(\rho)$ by Lemma 14, and this holds if and only if $FP(\rho)^{\perp} \cap FP(\beta)$ is a two-dimensional spacelike plane, i.e., if and only if $E(\rho, \beta)$ is homeomorphic with S^1.

Proof of (ii). If $\rho\beta \neq \beta\rho$, then $FP(\rho)^{\perp} \cap FP(\beta)$ is not two-dimensional by Lemma 14, but since $FP(\rho)^{\perp}$ and $FP(\beta)$ are two-dimensional subspaces of the time-zero plane, their intersection is one-dimensional and contains two opposite time-zero unit vectors. $\qquad\square$

2.6 Proof of Theorem 3

Let N^{e_0} be the set of all $(\tau, \beta) \in \ddot{R} \times B$ with $E(\tau, \beta) \cong \mathbb{Z}_2$ (cf. Prop. 17). Define a map $\lambda_1 : N^{e_0} \to L_1$ by $\lambda_1(\sigma, \beta) := \sigma^2 \beta$ and define $L_1^{e_0} := \lambda_1(N^{e_0})$. Furthermore, set $\mathbf{G}_L^{e_0} := \tilde{\lambda}^{-1}(L_1^{e_0})$.

For each $\rho \in \ddot{R}$, there is a unique time-zero unit vector $\mathbf{a}(\rho)$ with the property that ρ is a right-handed rotation with respect to $\mathbf{a}(\rho)$ by a rotation angle $\alpha(\rho)$ smaller than π. The functions $\mathbf{a}(\cdot)$ and $\alpha(\cdot)$ are continuous on $\ddot{R}$, and α has a continuous extension to a function from all of R onto the closed interval $[0, \pi]$, we denote this extension by α as well.

For each $\beta \in \dot{B}$, there exists a unique time-zero unit vector $\mathbf{b}(\beta)$ with respect to which β is a boost by a rapidity $\chi(\beta)$ greater than zero. The functions $\mathbf{b}$ and χ are continuous, and the function χ has a continuous extension to all of B with values in $\mathbb{R}^{\geq 0}$, which we denote by χ as well.

The functions $\tilde{\alpha} : \mathbf{G}_L \to [0, \pi]$ and $\tilde{\chi} : \mathbf{G}_L \to \mathbb{R}^{\geq 0}$ defined by $\tilde{\alpha}(g) := \alpha(\tilde{\rho}(g))$ and $\tilde{\chi}(g) := \chi(\tilde{\beta}(g))$ are continuous.

Lemma 18.

(i) *The polar decomposition $\hat{\rho} \times \hat{\beta} : L_1 \to R \times B$ is continuous.*

(ii) *The restriction of the group product in L_1 to $R \times B$ is a homeomorphism onto L_1.*

(iii) *N^{e_0} is a two-sheeted covering space of $L_1^{e_0}$ when endowed with the covering map λ_1.*

Proof of (i). The group product in L_1, the map $\mu \mapsto \mu^*$, and the square-root function are continuous, the map $\mu \mapsto \hat{\beta}(\mu) := \sqrt{\mu^* \mu}$ is continuous. Since

the map $\mu \mapsto \mu^{-1}$ is continuous as well, one concludes that $\mu \mapsto \hat{\rho}(\mu) := \mu\hat{\beta}(\mu)^{-1}$ is continuous.

Proof of (ii). The group product is continuous and inverse to the continuous polar decomposition. Since the group product is onto, so is the polar decomposition.

Proof of (iii). N^{e_0} is an open subset of $\ddot{R} \times B$, so it suffices to prove the corresponding statement for $\ddot{R} \times B$. So it remains to be shown that $\ddot{R}$ is a two-sheeted covering space when endowed with the covering map $\tau \mapsto \tau^2$. Continuity of this map follows from continuity of the group product. Conversely, each $\rho \in \ddot{R}$ has the two roots $[\mathbf{a}(\rho), \alpha(\rho/2)]$ and $[-\mathbf{a}(\rho), \pi - \alpha(\rho)/2]$, and since $\mathbf{a}$ and α are continuous maps, the square map has a continuous local inverse. $\square$

Lemma 19. *For each $g \in G_L^{e_0}$, there is a unique square root $\tilde{\tau}(g)$ of $\tilde{\rho}(g)$ with $g = \pi(\tilde{\tau}(g)\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e})$ for both $e \in E(\tilde{\tau}(g), \tilde{\beta}(g))$.*

Proof. If $e \in FP(\tilde{\beta}(g))$, then $\lambda(\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e}) = \tilde{\beta}(g)$. If $e \in FP(\tilde{\rho}(g))^{\perp}$, there are precisely two $a \in \bar{Z}$ with $\lambda(a, \bar{e}) = \tilde{\rho}(g)$. Namely, if $\tau_{\pm}$ are the two square roots of the rotation $\tilde{\rho}(g)$, then $a_{\pm} = (\tau_{\pm}\bar{e}, \bar{e})$ do the job.

Accordingly, if $e \in E(\tilde{\rho}(g), \tilde{\beta}(g)) = FP(\tilde{\rho}(g)) \cap FP(\tilde{\beta}(g))^{\perp}$, the non-equivalent pairs $\underline{m}^+$ and $\underline{m}^-$ defined by

$$\underline{m}^{\pm} := (\tau_{\pm}\bar{e}, \bar{e}) \circ (\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e}) = (\tau_{\pm}\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e})$$

satisfy $\lambda(\underline{m}^{\pm}) = \tilde{\lambda}(g)$. By Corollary 2.3, exactly one of them is contained in $\pi^{-1}(g)$. $\square$

Define a "polar decomposition" $\eta : \mathbf{G}_L^{e_0} \to N^{e_0}$ by $\eta(g) := (\tilde{\tau}(g), \tilde{\beta}(g))$. Evidently η is a bijection, and the diagram

$$
\begin{array}{ccc}
 & & \mathbf{G}_L^{e_0} \\
 & {\eta} \swarrow & \downarrow {\tilde{\lambda}} \\
N^{e_0} & \xrightarrow{\lambda_1} & L_1^{e_0}
\end{array}
\qquad (4)
$$

commutes. Next define the set

$$\mathbf{M}_L^{e_0} := \{(\tau\bar{e}, \beta^{-1/2}\bar{e}) : (\tau, \beta) \in N^{e_0}, \ e \in E(\tau, \beta)\},$$

and define a map $\lambda_2 : \mathbf{M}_L^{e_0} \to N^{e_0}$ by $\lambda_2(\underline{m}) := \eta(\pi(\underline{m}))$. Then the diagrams

$$
\begin{array}{ccc}
\mathbf{M}_L^{e_0} \xrightarrow{\ \pi\ } \mathbf{G}_L^{e_0} & (A) \\
\lambda_2 \downarrow \quad \eta \nearrow \quad \downarrow \tilde{\lambda} & \\
N^{e_0} \xrightarrow[\ \lambda_1\ ]{} L_1^{e_0} &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathbf{M}_L^{e_0} \xrightarrow{\ \pi\ } \mathbf{G}_L^{e_0} & (B) \\
\lambda_2 \downarrow \quad \lambda \searrow \quad \downarrow \tilde{\lambda} & \\
N^{e_0} \xrightarrow[\ \lambda_1\ ]{} L_1^{e_0} &
\end{array}
\qquad (5)
$$

commute. Define a continuous function $\mathbf{e} : R \times B \to H_1 \cap e_0^{\perp}$ by

$$
\mathbf{e}(\rho, \beta) := \frac{\mathbf{a}(\rho) \times \mathbf{b}(\beta)}{|\mathbf{a}(\rho) \times \mathbf{b}(\beta)|},
$$

where $\times$ denotes the vector product within the time-zero plane $e_0^{\perp}$.

Lemma 20.

(i) $\lambda_{e_0} := \lambda|_{\mathbf{G}_L^{e_0}}$ *is an open map.*

(ii) λ_2 *is continuous.*

(iii) η *is continuous.*

Proof of (i). $L_1^{e_0}$ is first-countable, so it suffices to show that for each sequence $(\mu_n)_n$ in $L_1^{e_0}$ converging to a limit μ and for each $\underline{m} \in \lambda_{e_0}^{-1}(\mu)$, there exists a sequence $(\underline{m}_n)_n$ converging to $\underline{m}$ and satisfying $\lambda_{e_0}(\underline{m}_n) = \mu_n$.

So let $(\mu_n)_n$ be a sequence in $L_1^{e_0}$ converging to μ. Then $\hat{\rho}(\mu_n)$ and $\hat{\beta}(\mu_n)$ converge to $\hat{\rho}(\mu)$ and $\hat{\beta}(\mu)$, respectively, by continuity of the functions $\hat{\rho}$ and $\hat{\beta}$. Consequently, the time-zero unit vectors $e_n := \mathbf{e}(\hat{\rho}(\mu_n), \hat{\beta}(\mu_n))$ tend to the limit $e = \mathbf{e}(\hat{\rho}(\mu), \hat{\beta}(\mu))$. Since $\bar{\pi}$ is continuous, the sequence $\bar{e}_n$ converges to $\bar{e}$.

Consider, without loss, the element $\underline{m} := (\tau\bar{e}, \hat{\beta}(\mu)^{-1/2}\bar{e})$ of the fiber $\lambda^{-1}(\mu)$. There exists a convergent sequence $(\tau_n)_n$ in R with $\tau_n^2 = \hat{\rho}(\mu_n)$, and the sequence $(\underline{m}_n)_n$ defined by $\underline{m}_n := (\tau_n\bar{e}_n, \hat{\beta}(\mu_n)^{-1/2}\bar{e}_n)$ satisfies $\lambda_{e_0}(\underline{m}_n) = \mu_n$ and $\underline{m}_n \to \underline{m}$. The same reasoning applies to the other elements of the fiber $\lambda_{e_0}^{-1}(\mu)$.

Proof of (ii). For each $\underline{m}_1 \in \mathbf{M}_L^{e_0}$, the fiber $\lambda_{e_0}^{-1}(\lambda_{e_0}(\underline{m}_1))$ contains four elements $\underline{m}_1, \ldots, \underline{m}_4$, and by the Hausdorff property, these have mutually disjoint open neighborhoods $U_1, \ldots, U_4$. Since λ_{e_0} is open by statement (i), their images are open, so $V := \lambda_{e_0}(U_1) \cap \cdots \cap \lambda_{e_0}(U_4)$ is open.

On the other hand, there is an open neighborhood Y of $\lambda_2(\underline{m}_1)$ with the property that $\lambda_1|_Y$ is a homeomorphism onto $W := \lambda_1(Y)$. Being a covering map, λ_1 is open, so W is open.

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$V \cap W$ is open, and λ_{e_0} is continuous, so the set $X := U_1 \cap \lambda_{e_0}^{-1}(V \cap W)$ is open and contains $\underline{m}_1$. The diagram

$$
\begin{array}{ccc}
X & & \\
\Big\downarrow {\scriptstyle \lambda_2|_X} & \searrow^{\ \lambda_{e_0}|_X} & \\
Y & \xrightarrow[\ \lambda_1|_Y\ ]{} & V \cap W
\end{array}
$$

is a commutative diagram of bijections by construction. Since $\lambda_{e_0}|_X$ and $\lambda_1|_Y$ are homeomorphims, so is $\lambda_2|_X$.

Proof of (iii). Using diagram 5 (B), one immediately concludes the statement from continuity of λ_2. $\qquad\square$

Lemma 21. $\mathbf{M}_L^{e_0}$ *is a two-sheeted covering space of N^{e_0} when endowed with the covering map λ_2.*

Proof. Define continuous maps $\underline{m}_\pm : N^{e_0} \to \mathbf{M}_L^{e_0}$ by

$$\underline{m}_\pm(\tau, \beta) := (\pm\tau\bar{e}(\tau, \beta), \pm\beta\bar{e}(\tau, \beta)).$$

We show that these functions are local inverses of λ_2.

For a given $x \in N^{e_0}$, write $y_\pm := \underline{m}_\pm(x)$. Since $\mathbf{M}_L^{e_0}$ is a Hausdorff space, there exist two disjoint open neighborhoods $Y_\pm$ of $y_\pm$. By continuity of $\underline{m}_\pm$, the pre-images $X_\pm := \underline{m}_\pm^{-1}(Y_\pm)$ are open, and $X := X_+ \cap X_-$ is an open neighborhood of x. By continuity of λ_2, the sets $W_\pm := \lambda_2^{-1}(X) \cap Y_\pm$ are open neighborhoods of $\underline{m}_\pm(x)$ with $\lambda_2(W_+) = X = \lambda_2(W_-)$. As a consequence, the continuous maps $\underline{m}_\pm|_X : U \to W_\pm$ are one-to-one and onto, their inverse being λ_2. $\qquad\square$

Proposition 22.

(i) η *is a homeomorphism.*

(ii) $\mathbf{G}_L^{e_0}$ *is a Hausdorff space.*

(iii) $\mathbf{G}_L^{e_0}$ *is a two-sheeted covering space of $L_1^{e_0}$ when endowed with the covering map $\tilde{\lambda}_{e_0}$.*

Proof. The maps $\pi \circ \underline{m}_+$ and $\pi \circ \underline{m}_-$ coincide and are inverse to η by construction. By continuity of $\underline{m}_\pm$ and π, they are continuous. This proves (i) and implies (ii).

$\tilde{\lambda}_{e_0} = \lambda_2 \circ \eta$ is a concatenation of a homeomorphism and a two-sheeted covering map. This yields (iii). $\qquad\square$

Next we extend these results to $\dot{\mathbf{G}}_L$. To this end, recall that $\mu \in L_1^{e_0}$ if and only if $\hat{\rho}(\mu)\hat{\beta}(\mu) \neq \hat{\beta}(\mu)\hat{\rho}(\mu)$.

Proposition 23.

(i) *For each $e_0 \in M_1^+$, the set $\mathbf{G}_L^{e_0}$ is an open subset of $\dot{\mathbf{G}}_L$.*

(ii) $\bigcup_{e_0 \in M_1^+} \mathbf{G}_L^{e_0} = \dot{\mathbf{G}}_L$.

(iii) $\dot{\mathbf{G}}_L$ *is a two-sheeted covering space of $L_1 \backslash \{1\}$ when endowed with the covering map $\tilde{\lambda}$.*

Proof. If a sequence $\mu_n \to \mu$ in L_1 with $\hat{\rho}(\mu_n)\hat{\beta}(\mu_n) = \hat{\beta}(\mu_n)\hat{\rho}(\mu_n)$, then $\hat{\rho}(\mu)\hat{\beta}(\mu) = \hat{\beta}(\mu)\hat{\rho}(\mu)$. Namely, one has $\hat{\beta}(\mu_n)^{-1}\hat{\rho}(\mu_n)\hat{\beta}(\mu_n)\hat{\rho}(\mu_n)^{-1} = 1$ for all n, so $\hat{\beta}(\mu)^{-1}\hat{\rho}(\mu)\hat{\beta}(\mu)\rho^{-1}(\mu) = 1$ follows by continuity of the functions β, ρ, $\hat{\beta}(\cdot)^{-1}$, and $\hat{\rho}(\cdot)^{-1}$, and of the group product.

As a consequence, the set $L_1^{e_0}$ has a closed complement and, hence, is an open subset of L_1. Accordingly, $\mathbf{G}_L^{e_0} = \tilde{\lambda}^{-1}(L_1^{e_0})$ is open by continuity of $\tilde{\lambda}$. This proves (i).

It follows from Lemma 16 that $\bigcup_{e_0 \in M_1^+} L_1^{e_0} = L_1 \backslash \{1\}$, and this proves statement (ii) by continuity of $\tilde{\lambda}$.

By statements (i) and (ii), there is, for each $g \in \dot{\mathbf{G}}_L$, an open neighborhood restricted to which $\tilde{\lambda}$ is one-to-one and open. This proves (iii). $\square$

Proposition 24.

(i) $\mathbf{G}_L$ *is a Hausdorff space.*

(ii) $\tilde{\lambda}$ *is open.*

Proof of (i). Being a union of Hausdorff spaces, $\dot{\mathbf{G}}_L$ is a Hausdorff space, so it remains to prove that for each g there are disjoint neighborhoods U_1 and U_g of 1 and $g \neq 1$, respectively, (which implies that there are disjoint neighborhoods $-U_1$ and $-U_g$ of -1 and $-g$).

$g \neq 1$ implies that $(\tilde{\alpha}(g), \tilde{\chi}(g)) \neq (0,0)$. Since $\tilde{\alpha}$ and $\tilde{\chi}$ are continuous[4] and since $(\tilde{\alpha}(h), \tilde{\chi}(h)) = (0,0)$ implies $h = 1$, the open sets

$$U_1 := (\tilde{\alpha} \times \tilde{\chi})^{-1}(\qquad [0, \varepsilon) \qquad \times \qquad [0, \varepsilon) \qquad)$$
$$\text{and } U_g := (\tilde{\alpha} \times \tilde{\chi})^{-1}(\quad (\tilde{\alpha}(g) - \varepsilon, \tilde{\alpha}(g) + \varepsilon) \quad \times \quad (\tilde{\chi}(g) - \varepsilon, \tilde{\chi}(g) + \varepsilon) \quad)$$

are disjoint for sufficiently small $\varepsilon > 0$.

---

[4]with respect to the *relative* topologies of the *closed* topological subspaces $[0, \pi]$ and $\mathbb{R}^{\geq 0}$ of $\mathbb{R}$,

Proof of (ii). It has been shown that $\dot{\mathbf{G}}_L$ is a two-sheeted covering space when endowed with the covering map $\tilde{\lambda}$. Since $\tilde{\lambda}$ is continuous on all of $\mathbf{G}_L$, it remains to be shown that $\tilde{\lambda}$ is open at ± 1. L_1 is first countable, so it suffices to show that for each sequence $\mu_n \to 1$ in L_1 there exists a sequence $g_n \to 1$ in $\mathbf{G}_L$ with $\tilde{\lambda}(g_n) = \mu_n$; note that the sequence $(-g_n)_n$ tends to -1 in this case.[5] For each n there is a $g_n \in \tilde{\lambda}^{-1}(\mu_n)$ with $\tilde{\alpha}(g_n) \leq \pi/2$. For any $\varepsilon > 0$, almost all g_n satisfy $(\tilde{\alpha}(g_n), \tilde{\chi}(g_n)) \in [0, \pi] \times [0, \varepsilon]$. Since this is a compact set, the sequence $(\tilde{\alpha}(g_n), \tilde{\chi}(g_n))$ has at least one accumulation point. $\tilde{\beta}(g_n)$ tends to 1, so $\tilde{\chi}(g_n)$ tends to zero, so all accumulation points are in $[0, \pi] \times \{0\}$.

The assumption $\mu_n \to 1$ further reduces the set of possible points to $\{(0, 0), (\pi, 0)\}$, and opting for $\tilde{\alpha}(g_n) \leq \pi/2$ rules out $(\pi, 0)$. So both $\tilde{\alpha}(g_n)$ and $\tilde{\chi}(g_n)$ tend to zero. It follows that g_n tends to 1. $\qquad\square$

We now recall and prove the remaining statements made in Sect. 1.2.

Theorem 3.(i). $\mathbf{G}_L$ *is a two-sheeted covering space of L_1 when endowed with the covering map $\tilde{\lambda}$.*

Proof. $\dot{\mathbf{G}}_L$ is a covering of $L_1 \backslash \{1\}$ when endowed with the covering map $\tilde{\lambda}$, so all that remains to be shown is that $\tilde{\lambda}$ is a homeomorphism from some neighborhood U of 1 or -1 onto $\tilde{\lambda}(U)$.

Since $\mathbf{G}_L$ is a Hausdorff space, there exist disjoint neighborhoods $U_\pm$ of ± 1. Since $\tilde{\lambda}$ is open, the images $V_\pm := \tilde{\lambda}(U_\pm)$ are open. The intersection $V := V_+ \cap V_-$ is an open neighborhood of $1 \in L_1$, and by continuity of $\tilde{\lambda}$, the sets $W_\pm := U \cap \tilde{\lambda}^{-1}(V_+ \cap V_-)$ are open and, hence, neighborhoods of $\pm 1 \in \mathbf{G}_L$, respectively. Since $W_\pm$ have been constructed in such a fashion that $\tilde{\lambda}(W_+) = U = \tilde{\lambda}(W_-)$, the restrictions $\tilde{\lambda}_\pm$ to $W_\pm$ are one-to-one and onto, and since $\tilde{\lambda}$ is open, the inverse mappings are continuous. $\qquad\square$

Theorem 3.(ii). $\mathbf{G}_L$ *is simply connected.*

Proof. $\bar{Z}$ is pathwise connected, so $\mathbf{M}_L = \bar{Z} \times \bar{Z}$ is pathwise connected, and since π is continuous, $\mathbf{G}_L = \pi(\mathbf{M}_L)$ is pathwise connected. Since $\mathbf{G}_L$ is a two-sheeted covering group of L_1, and since the fundamental group of L_1

---

[5]It suffices to consider sequences, since L_1 is first-countable (which we have not yet been proved for $\mathbf{G}_L$ at this stage). Namely, let $U_g \subset \mathbf{G}_L$ be a neighborhood of any $g \in \mathbf{G}_L$, and let $(\mu_n)_n$ be a sequence in L_1 converging to $\tilde{\lambda}(g)$. By assumption there is a sequence $g_n \to G$ with $\tilde{\lambda}(g_n) \to \mu_n$. Since $g_n \to g$ and since U_g is a neighborhood of g, one has $g_n \in U_g$ for almost all n, so $\mu_n = \tilde{\lambda}(g_n) \in \tilde{\lambda}(U_g)$ for almost all n. Since this holds for all sequences $\mu_n \to \tilde{\lambda}(g)$ one concludes that $\tilde{\lambda}(U_g)$ is a neighborhood of $\tilde{\lambda}(g)$ in L_1 by first-countability.

is isomorphic with $\mathbb{Z}_2$, one concludes that $\mathbf{G}_L$ is homeomorphic with the universal covering of L_1. $\qquad\square$

Theorem 3.(iii). *There is a unique group product $\odot$ on $\mathbf{G}_L$ with the property that the diagram*

$$
\begin{array}{ccc}
\mathbf{M}_L \times \mathbf{M}_L & \xrightarrow{\;\circ\;} & \mathbf{M}_L \\
{\scriptstyle \pi\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\mathbf{G}_L \times \mathbf{G}_L & \xrightarrow{\;\odot\;} & \mathbf{G}_L \\
{\scriptstyle \tilde\lambda\times\tilde\lambda}\big\downarrow & & \big\downarrow{\scriptstyle \tilde\lambda} \\
L_1 \times L_1 & \xrightarrow{\;\cdot\;} & L_1
\end{array}
\tag{6}
$$

commutes.

Proof. The outer arrows of the diagram commute, so it suffices to prove existence and uniqueness of a group product conforming with the lower part. But it is well known that each simply connected covering space $\tilde{G}$ of a topological group G can be endowed with a unique group product $\odot$ such that G is a covering group.[6] $\qquad\square$

Lemma 4. *Given $h \in \mathbf{G}_L$ and $(c,d) \in \mathbf{M}_L$, one has*

$$
h\pi(c,d)h^{-1} = \pi\left(\tilde\lambda(h)c, \tilde\lambda(h)d \right).
\tag{7}
$$

Proof. The function $F : G_L \to \mathbf{G}_L$ defined by

$$
F(h) := \pi\left(\tilde\lambda(h)c, \tilde\lambda(h)d \right)^{-1} h\pi(c,d)h^{-1}
$$

has the property that $\lambda(F(h)) = 1$ and that, hence, it takes values in the discrete set $\{\pm 1\} \subset \mathbf{G}_L$. Since F is continuous and L_1 is connected, F is constant, and because $F(1) = 1$, it follows that $F(h) = 1$ for all h. $\qquad\square$

3 Spin & Statistics

The preceding section has provided the basis of a general spin-statistics theorem, which is the subject of this section. From an intrinsic form of symmetry under a charge conjugation combined with a time inversion and the reflection in *one* spatial direction, which is referred to as *modular P_1CT-symmetry*, a strongly continuous unitary representation $\tilde{W}$ of $\mathbf{G}_L$ will be

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[6]See, e.g., Props. 5 and 6 in Sect. I.VIII. in Ref. 7.

constructed. It is, then, elementary to show that $\tilde{W}$ exhibits Pauli's spin-statistics relation.

Let F be an arbitrary quantum field on $\mathbb{R}^{1+3}$ in a Hilbert space $\mathcal{H}$. The standard properties of relativistic quantum field to be used here are practically the same as in Ref. 14 and are recalled here for the reader's convenience.

(A) *Algebra of field operators.* Let $\mathfrak{C}$ be a linear space of arbitrary dimension,[7] and denote by $\mathfrak{D}$ the space $C_0^\infty(\mathbb{R}^{1+3})$ of test functions on $\mathbb{R}^{1+3}$. The field F is a linear function that assigns to each $\Phi \in \mathfrak{C} \otimes \mathfrak{D}$ a linear operator $F(\Phi)$ in a separable Hilbert space $\mathcal{H}$.

(A.1) F is free from redundancies in $\mathfrak{C}$, i.e., if $\mathfrak{c}, \mathfrak{d} \in \mathfrak{C}$ and if $F(\mathfrak{c} \otimes \varphi) = F(\mathfrak{d} \otimes \varphi)$ for all $\varphi \in \mathfrak{D}$, then $\mathfrak{c} = \mathfrak{d}$.

(A.2) Each field operator $F(\Phi)$ and its adjoint $F(\Phi)^\dagger$ are densely defined. There exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ contained in the domains of $F(\Phi)$ and $F(\Phi)^\dagger$ and satisfying $F(\Phi)\mathcal{D} \subset \mathcal{D}$ and $F(\Phi)^\dagger\mathcal{D} \subset \mathcal{D}$ for all $\Phi \in \mathfrak{C} \otimes \mathfrak{D}$.

Denote by $\mathbf{F}$ the algebra generated by all $F(\Phi)|_\mathcal{D}$ and all $F(\Phi)^\dagger|_\mathcal{D}$. Defining an involution $*$ on $\mathbf{F}$ by $A^* := A^\dagger|_\mathcal{D}$, the algebra $\mathbf{F}$ is endowed with the structure of a $*$-algebra.

Let $\mathbf{F}(a)$ be the algebra generated by all $F(\mathfrak{c} \otimes \varphi)|_\mathcal{D}$ and all $F(\mathfrak{c} \otimes \varphi)^\dagger|_\mathcal{D}$ with $\mathrm{supp}(\varphi) \subset W_a$, where W_a denotes, as above, the Rindler wedge of a. The algebra $\mathbf{F}(a)$ inherits the structure of a $*$-algebra from $\mathbf{F}$ by restriction of $*$.

(A.3) $\mathbf{F}(a)$ is nonabelian for each a, and $a \neq b$ implies $\mathbf{F}(a) \neq \mathbf{F}(b)$.

(B) *Cyclic vacuum vector.* There exists a vector $\Omega \in \mathcal{D}$ that is cyclic with respect to each $\mathbf{F}(a)$.

(C) *Normal commutation relations.* There exists a unitary and self-adjoint operator k on $\mathcal{H}$ with $k\Omega = \Omega$ and with $k\mathbf{F}(a)k = \mathbf{F}(a)$ for all a. Define $F_\pm := \frac{1}{2}(F \pm kFk)$. If $\mathfrak{c}$ and $\mathfrak{d}$ are arbitrary elements of $\mathfrak{C}$ and if $\varphi, \psi \in \mathfrak{D}$ have spacelike separated supports, then

$$F_+(\mathfrak{c} \otimes \varphi)F_+(\mathfrak{d} \otimes \psi) = F_+(\mathfrak{d} \otimes \psi)F_+(\mathfrak{c} \otimes \varphi),$$
$$F_+(\mathfrak{c} \otimes \varphi)F_-(\mathfrak{d} \otimes \psi) = F_-(\mathfrak{d} \otimes \psi)F_+(\mathfrak{c} \otimes \varphi), \quad \text{and}$$
$$F_-(\mathfrak{c} \otimes \varphi)F_-(\mathfrak{d} \otimes \psi) = -F_-(\mathfrak{d} \otimes \psi)F_-(\mathfrak{c} \otimes \varphi)$$

---

[7]Again, $\mathfrak{C}$ is the "component space", and its dimension equals the number of components, which may be infinite in what follows.

for all $\mathfrak{c}, \mathfrak{d} \in \mathfrak{C}$.

The involution k is the *statistics operator*, and $F_\pm$ are the bosonic and fermionic components of F, respectively. Defining $\kappa := (1 + ik)/(1 + i)$ and $F^t(\mathfrak{d} \otimes \psi) := \kappa F(\mathfrak{d} \otimes \psi)\kappa^\dagger$, the normal commutation relations read

$$\left[F(\mathfrak{c} \otimes \varphi), F^t(\mathfrak{d} \otimes \psi)\right] = 0.$$

This property is referred to as *twisted locality*. Denote $\mathbf{F}(a)^t := \kappa \mathbf{F}(a)\kappa^\dagger$.

These properties imply that Ω is *separating* with respect to each algebra $\mathbf{F}(a)$, i.e., there is no nonzero operator $A \in \mathbf{F}(a)$ with $A\Omega = 0$.[8] As a consequence, an antilinear operator $R_a : \mathbf{F}(a)\Omega \to \mathbf{F}(a)\Omega$ is defined by $R_a A\Omega := A^*\Omega$. This operator is closable. Its closed extension S_a has a unique polar decomposition $S_a = J_a \Delta_a^{1/2}$ into an antiunitary operator J_a, which is called the *modular conjugation*, and a positive operator $\Delta_a^{1/2}$, which is called the *modular operator*. J_a is an involution.[9]

For each $a \in \bar{Z}$, let j_a be the orthogonal reflection at the edge of W_a.

(D) *Modular P_1CT-symmetry.* For each $a \in \bar{Z}$, there exists a linear involution C_a in $\mathfrak{C}$ such that for all $\mathfrak{c} \in \mathfrak{C}$ and $\varphi \in \mathfrak{D}$, one has

$$J_a F(\mathfrak{c} \otimes \varphi)J_a = F^t(C_a\mathfrak{c} \otimes \overline{j_a\varphi}).$$

The map $a \mapsto J_a$ is strongly continuous.[10]

It will now be recalled that pairs of modular P_1CT-reflections give rise to a strongly continuous representation of $\mathbf{G}_L$ which exhibits Pauli's spin-statistics connection.

Lemma 25. *Let K be a unitary or antiunitary operator in $\mathcal{H}$ with $K\Omega = \Omega$, and suppose there are $a, b \in \bar{Z}$ such that $K\mathbf{F}(a)K^\dagger = \mathbf{F}(b)$. Then $KJ_aK^\dagger = J_b$, and $K\Delta_a K^\dagger = \Delta_b$.*

Proof. If $B \in \mathbf{F}(b)$, then $KS_a K^\dagger B\Omega = KS_a \underbrace{K^\dagger BK}_{\in \mathbf{F}(a)}\Omega = B^*\Omega = S_b B\Omega$. The statement now follows by uniqueness of the polar decomposition. $\square$

---

[8]For details on this and the following statements, see Ref. 14 or 3.

[9]S_a, J_a, and $\Delta_a^{1/2}$ are the objects of the well-known *modular theory* developed by Tomita and Takesaki.

[10]If one assumes covariance with respect to some strongly continuous representation of $\mathbf{G}_L$ (which may also violate the spin-statistics connection), this is straightforward to derive. But covariance, as such, is not needed.

This lemma yields a couple of important relations. For each $a \in \bar{Z}$, one has $k\mathbf{F}(a)k^{\dagger} = k\mathbf{F}(a)k = \mathbf{F}(a)$, so

$$kJ_ak = J_a, \quad \text{whence} \quad J_a\kappa = \kappa^{\dagger}J_a \tag{8}$$

follows by antilinearity of J_a. By modular P_1CT-symmetry, $J_a\mathbf{F}(a)J_a = \mathbf{F}^t(-a) = \kappa\mathbf{F}(-a)\kappa^{\dagger}$, so

$$J_a = J_aJ_aJ_a = \kappa J_{-a}\kappa^{\dagger}. \tag{9}$$

It also follows from modular P_1CT-symmetry that $J_a\mathbf{F}(b)J_a = \mathbf{F}^t(j_ab) = \mathbf{F}(-j_ab)$, so

$$J_aJ_bJ_a = J_{-j_ab} = J_{j_aj_bb} = J_{\lambda(a,b)b}. \tag{10}$$

These consequences of Lemma 25 will be used extensively in what follows without further mentioning.

For every $(a, b) \in \mathbf{M}_L$, define $W(a, b) := J_aJ_b$.

Theorem 26.

(i) *There exists a representation $\tilde{W} : \mathbf{G}_L \to W(\mathbf{M}_L)$ in $\mathcal{H}$ with the property that the diagrams*

$$\tag{11}$$

   commute.

(ii) *There is a representation $\tilde{D}$ of $\mathbf{G}_L$ in $\mathfrak{C}$ such that*

$$\tilde{W}(g)F(\mathfrak{c} \otimes \varphi)\tilde{W}(g)^* = F(\tilde{D}(g)\mathfrak{c} \otimes \tilde{\lambda}(g)\varphi) \quad \text{for all } g, \mathfrak{c}, \varphi, \tag{12}$$

   where $\tilde{\lambda}(g)\varphi := \varphi(\tilde{\lambda}(g)^{-1}\cdot)$.

Proof of (i). Fix some $e_0 \in M_1^+$.

For each $r \in \mathbf{G}_L$ with $\tilde{\lambda}(r) \in R$, there exists a unique rotation τ with $\tau^2 = \tilde{\lambda}(r)$ and $r = \pi(\tau\bar{e}, \bar{e})$ for all time-zero unit vectors in $FP(\lambda(r))^\perp$. For each $\underline{n} := (\bar{e}', \bar{f}') \sim \underline{m}$, there exists a rotation ρ with $\rho\lambda(\underline{m})\rho^{-1} = \lambda(\underline{m})$ and $\rho^2\underline{m} = \underline{n}$. Because $a \mapsto J_a$ and, hence, also the map W is continuous by assumption of modular P_1CT-symmetry, one can mimick the proof of Lemma 2.4 in Ref. 6 in order to show that $W(\underline{m}) = W(\underline{n})$, and one can define a unitary operator $\tilde{W}_{e_0}(r)$ by $\tilde{W}_{e_0}(r) := W(\underline{m})$. It has been shown in Ref. 14 that these operators give rise to a representation of the subgroup $\mathbf{G}_R := \tilde{\lambda}^{-1}(R)$ of $\mathbf{G}_L$.

For each $b \in \mathbf{G}_L$ with $\tilde{\lambda}(b) \in B$, the class $\pi^{-1}(b)$ contains elements of the form either $(\bar{e}, \beta\bar{e})$ or $(\bar{e}, -\beta\bar{e})$, where $\beta := \tilde{\beta}(b)^{-1/2}$. The one-parameter group of rotations around the boost direction of β acts transitively on the set of such elements. If $\underline{n}$ is a second such argument equivalent to $\underline{m}$, one can, again, use the reasoning of Ref. 6 in order to show that $W(\underline{m}) = W(\underline{n})$, and one can define $\tilde{W}_{e_0}(b) := W(\underline{m})$. Furthermore, if $(b_t)_t$ is a one-parameter subgroup of $\mathbf{G}_L$ with $\tilde{\lambda}(b_t) \in B$, it follows from the results of Ref. 6 that $\tilde{W}(b_s)\tilde{W}(b_t) = \tilde{W}(b_{s+t})$.

The polar decomposition in L_1 can be lifted to a polar decomposition in $\mathbf{G}_L$. Namely, given an arbitrary $g \in \mathbf{G}_L$, there exist $r_g, b_g \in \mathbf{G}_L$ with $\tilde{\lambda}(r_g) = \hat{\rho}(\tilde{\lambda}(g))$ and $\tilde{\lambda}(b_g) = \hat{\beta}(\tilde{\lambda}(g))$ and $r_g b_g = g$. This decomposition is unique up to replacement of r_g and b_g by $-r_g$ and $-b_g$, respectively.

Therefore, the operator $\tilde{W}(g) := \tilde{W}(r_g)\tilde{W}(b_g)$ does not depend on the choice of this polar decomposition.

For arbitrary $g = r_g b_g \in \mathbf{G}_L$, define $\tilde{W}_{e_0}(g) := \tilde{W}_{e_0}(r_g)\tilde{W}_{e_0}(b_g)$. Note that the definition of $\tilde{W}_{e_0}(g)$ depends on e_0 as it stands; but the index will be dropped for the time being.

Lemma 27. *Consider $g, h \in \mathbf{G}_L$ with $\tilde{\lambda}(g), \tilde{\lambda}(hgh^{-1}) \in B$ and $\tilde{\lambda}(h) \in R \cup B$. Then*

$$\tilde{W}(h)\tilde{W}(g)\tilde{W}(h)^* = \tilde{W}\left(hgh^{-1}\right).$$

Proof. It follows from eq. (10) and Lemma 4 that for each $(a, b) \in \pi^{-1}(g)$, one has

$$\begin{aligned}
\tilde{W}(h)\tilde{W}(\pi(a,b))\tilde{W}(h)^* = \tilde{W}(h)J_a J_b \tilde{W}(h)^* &= J_{\tilde{\lambda}(h)a}J_{\tilde{\lambda}(h)b} \\
&= W(\tilde{\lambda}(h)a, \tilde{\lambda}(h)b) = \tilde{W}\left(\pi\left(\tilde{\lambda}(h)a, \tilde{\lambda}(h)b\right)\right) \quad (13) \\
&= \tilde{W}(h\pi(a,b)h^{-1}).
\end{aligned}$$

$\square$

Lemma 28. *If $\tilde{\lambda}(g) \in \mathfrak{S}(a)$ for some $a \in \bar{Z}$, then $W(\underline{m}) = \tilde{W}(g)$ for all $\underline{m} \in \pi^{-1}(g)$.*

Proof. Without loss, suppose that $a = \bar{e}$ for some time-zero unit vector e. If $g = r_g b_g$ and $h = r_h b_h$ with $\tilde{\lambda}(g), \tilde{\lambda}(h) \in \mathfrak{S}(\bar{e})$ for some time-zero unit vector e, then Lemma 27 implies

$$\tilde{W}(h)\tilde{W}(g)\tilde{W}(h)^* = \tilde{W}(r_h)\tilde{W}(b_h) \cdot \tilde{W}(r_g)\tilde{W}(b_g) \cdot \tilde{W}(b_h)^*\tilde{W}(r_h)^* \\ = \tilde{W}(g) \tag{14}$$

Let $\underline{m} \in \mathbf{M}_L$ satisfy $W(\underline{m}) = \tilde{W}(g)$. If $\underline{n} \sim \underline{m}$ and $\underline{n} \neq \underline{m}$, then there exists, by definition of $\sim$, a $\mu \in L_1$ with $\mu^2 \neq 1$, commuting with $\tilde{\lambda}(g)$ and satisfying $\mu^2 \underline{m} = \pm\underline{n}$. Since $\mathfrak{S}(\bar{e})$ is a maximal abelian group and since $\tilde{\lambda}(g) \in \mathfrak{S}(\bar{e})$ by assumption, one concludes $\mu \in \mathfrak{S}(\bar{e})$, and for each h with $\tilde{\lambda}(h) = \mu$, one obtains from eq. (14)

$$W(\underline{n}) = W(\pm\mu^2\underline{m}) = W(\mu^2\underline{m}) = \tilde{W}(h)\tilde{W}(g)\tilde{W}(h)^* = \tilde{W}(g) = W(\underline{m}).$$

$\square$

Proof of (i) (contd.). Next let $g \in \mathbf{G}_L$ be arbitrary with polar decomposition $r_g b_g$.

$\tilde{W}(g)$ is an element of $W(\mathbf{M}_L)$. Namely, recall that there exist a time-zero unit vector e and a rotation τ such that $g = \pi(\tau\bar{e}, \beta^{-1/2}\bar{e})$, where $\beta = \tilde{\lambda}(b_g) \in B$. One concludes

$$\tilde{W}(g) = \tilde{W}(r)\tilde{W}(b) = J_{\tau\bar{e}} J_{\bar{e}} J_{\bar{e}} J_{\beta^{-1/2}\bar{e}} \\ = J_{\tau\bar{e}} J_{\beta^{-1/2}\bar{e}} \in W(\mathbf{M}_L)$$

$\tilde{W}$ is a representation. Namely,

$$\tilde{W}(g)\tilde{W}(h) = \tilde{W}(r_g)\tilde{W}(b_g)\tilde{W}(r_h)\tilde{W}(b_h) \\ = \tilde{W}(r_g)\tilde{W}(r_h)\left(\tilde{W}(r_h)^*\tilde{W}(b_g)\tilde{W}(r_h)\right)\tilde{W}(b_h) \\ =: \tilde{W}(r_g r_h)\,\tilde{W}(b_f)\tilde{W}(b_h)$$

The last two terms implement the generalized boost

$$\tilde{\lambda}(b_f)\tilde{\lambda}(b_h) = \tilde{\lambda}(b_f)^{1/2}\left(\tilde{\lambda}(b_f)^{1/2}\tilde{\lambda}(b_g)\tilde{\lambda}(b_f)^{1/2}\right)\tilde{\lambda}(b_f)^{-1/2},$$

so Lemma 28 yields

$$\tilde{W}(b_f b_h) = J_{\pm\tilde{\lambda}(b_f)^{1/2}\bar{e}} J_{\tilde{\lambda}(b_h)^{-1/2}\bar{e}}$$

$$= J_{\pm\tilde{\lambda}(b_f)^{1/2}\bar{e}} J_{\bar{e}}^2 J_{\tilde{\lambda}(b_h)^{-1/2}\bar{e}} = J_{\bar{e}} J_{\pm\tilde{\lambda}(b_f)^{-1/2}\bar{e}} J_{\bar{e}} J_{\tilde{\lambda}(b_h)^{-1/2}\bar{e}}$$

$$= \tilde{W}(b_f)\tilde{W}(b_h).$$

Now write $b_f b_h =: d = r_d b_d$, then

$$\tilde{W}(g)\tilde{W}(h) = \tilde{W}(r_g r_h r_d)\tilde{W}(b_d)$$

$$= \tilde{W}(r_g r_h r_d b_d)$$

$$= \tilde{W}(gh).$$

Proof of (ii). Define a map D from $\mathbf{M}_L$ into the automorphism group $\operatorname{Aut}(\mathfrak{C})$ of $\mathfrak{C}$ by $D(a,b) := C_a C_b$. If $(a,b) \sim (c,d)$, then modular $\mathrm{P}_1\mathrm{CT}$-symmetry implies

$$F(C_a C_b \mathfrak{c} \otimes j_a j_b \varphi) = W(a,b) F(\mathfrak{c} \otimes \varphi) W(a,b)^*$$

$$= W(c,d) F(\mathfrak{c} \otimes \varphi) W(c,d)^*$$

$$= F(C_c C_d \mathfrak{c} \otimes j_c j_d \varphi)$$

$$= F(C_c C_d \mathfrak{c} \otimes j_a j_b \varphi)$$

for all $\mathfrak{c}$ and all φ. Using assumption (A.1), one obtains $C_a C_b \mathfrak{c} = C_c C_d \mathfrak{c}$ for all $\mathfrak{c}$, so $D(a,b) = D(c,d)$, and a map $\tilde{D} : \mathbf{G}_L \to \operatorname{Aut}(\mathfrak{C})$ is defined by $\tilde{D}(\pi(\underline{m})) := D(\underline{m})$. This map $\tilde{D}$ now inherits the representation property from $\tilde{W}$. $\qquad\square$

Theorem 29 (Spin-statistics connection).

$$F_\pm(\mathfrak{c} \otimes \varphi) = \frac{1}{2}(1 \pm F(\tilde{D}(-1)\mathfrak{c} \otimes \varphi)$$

for all $\mathfrak{c}$ and all φ.

Proof. For each $a \in \bar{Z}$ one has

$$\tilde{W}(-1) = J_a J_{-a} = J_a \kappa J_a \kappa^\dagger = J_a^2 (\kappa^\dagger)^2 = k,$$

so

$$kF(\mathfrak{c} \otimes \varphi)k = \tilde{W}(-1) F(\mathfrak{c} \otimes \varphi) \tilde{W}(-1)$$

$$= \tilde{W}(-1) F(\mathfrak{c} \otimes \varphi) \tilde{W}(-1)^\dagger = \tilde{F}(\tilde{D}(-1)\mathfrak{c} \otimes \varphi). \qquad\square$$

If, in particular, $\tilde{D}$ is irreducible with spin s, then $\tilde{D}(-1) = e^{2\pi i s}$, so $F_- = 0$ for integer s and $F_+ = 0$ for half-integer s.

4   Other modular symmetries

Evidently, the operator $\Theta := J_{\bar{e}_1} J_{\bar{e}_2} J_{\bar{e}_3}$ implements a full PCT-symmetry. Θ depends on the handedness of the triple (e_1, e_2, e_3) only [14].

As mentioned earlier, Guido and Longo obtained a spin-statistics theorem in the above spirit in Ref. 10. Instead of the P_1CT-reflections, they assumed the modular groups associated with the algebras $\mathbf{F}(a)$ and the vacuum vector, which satisfy the KMS-condition, to implement Lorentz boosts — which is the abstract verson of the Unruh effect. This suffices to construct a representation of L_1 not from P_1CT-reflections, but from the one-parameter groups implementing the boosts (for which the commutation relations requested for covariance are not assumed from the outset [4, 10]). This representation can, then easily be shown to satisfy Pauli's spin-statistics relation.

Since both the above and their representation have been constructed from the basic elements of Tomita-Takesaki theory, one should expect them to coincide. Indeed, this is he case.

Denote by Λ_a the unique one-parameter group of Lorentz boosts that map the wedge W_a onto itself for each $a \in \bar{Z}$. F exhibits the Unruh effect if and only if for each $a \in \bar{Z}$ there exists a one-parameter group V' of internal symmetries of F with $\Delta_a^{it} F(x) \Delta_a^{-it} = V'(\Lambda_a(t)) F(\Lambda_a(-t)x)$. In this case the modular unitaries Δ_a^{it}, $a \in \hat{Z}$, $t \in \mathbb{R}$, generate a covariant unitary representation U' of $\widetilde{L}_1$ [4]. This representation exhibits Pauli's spin-statistics connection [10].

On the other hand, the Unruh effect implies modular P_1CT-symmetry [10], and, hence, yields the representation W constructed above as well.

Lemma 30. $U' = W$.

Proof. It suffices to show that for each Δ_a^{it}, $a \in \hat{Z}$, $t \in \mathbb{R}$, there exist $b, c \in \hat{Z}$ such that $\Delta_a^{it} = J_b J_c$.

If for some $a, b \in \hat{Z}$ there exist zweibeine $\xi \in \bar{\pi}^{-1}(a)$ and $\eta \in \bar{\pi}^{-1}(b)$ with $t_\xi = t_\eta$ and $x_\xi \perp x_\eta$, then it follows from Lemma 25 that $J_a \Delta_a^{it} J_a = \Delta_a^{-it}$ and $\Delta_a^{-it/2} J_a \Delta_a^{it/2} = J_{\Lambda_a(t/2)}$ for all $t \in \mathbb{R}$, so

$$J_a \Delta_a^{it} = \Delta_a^{-it/2} J_a \Delta^{it/2} = J_{\Lambda_a(-t/2)a}, \quad \text{i.e.,} \quad \Delta_a^{it} = J_a J_{\Lambda_a(-t/2)a}. \qquad \square$$

Conclusion

Both the classical geometry and the fundamental quantum field theoretic representations of the restricted Lorentz group L_1 are based on reflection

symmetries. At the classical level, a simply connected covering group $\mathbf{G}_L$ of L_1 can be constructed from $P_1 T$-reflections.

For a typical quantum field F, a class of antiunitary $P_1 CT$-operators exists that are fixed by the intrinsic structure of the respective field. These are the fundamental symmetries of quantum field theories, and they give rise to a unitary representation of the Lorentz group. In order to show this, the existence of such a representation does not need to be assumed from the outset. On the other hand, the construction yields a distinguished representation of the Lorentz group even in cases where several covariant representations are present.

It may happen in such cases that representations satisfying Pauli's relation coexist with representations violating it. In any case, the representation constructed from the modular $P_1 CT$-conjugations exhibits the right spin-statistics connection, and this is, eventually, straightforward to see.

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